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CONSTRUCTING UNCONDITIONAL FINITE DIMENSIONAL DECOMPOSITIONS

BY

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ABSTRACT

Primariness of a Banach space is almost always obtained through the use of the Pelczynski decomposition method. In this paper we show that it is possible to directly construct UFDD's in many cases from which the primariness can be deduced. We give applications to l_p and X_p.

0. Introduction

A basic problem in Banach space theory is to determine which properties of a Banach space are inherited by its complemented subspaces. Reflexivity is an obvious example of a property which is inherited. On the other hand Szarek [Sz] has shown that there is a complemented subspace of a space with a basis which fails to have a basis. In this paper we show that complemented subspaces of spaces with UFDD's (unconditional finite dimensional decompositions) satisfying certain extra assumptions have a UFDD. The techniques may have wide application, however finding an appropriate setting in which to use the techniques seems to be difficult in some cases. We are however able to do this in many cases where the Pelczynski decomposition method has been used in the past and, in addition, to obtain a localization of a result of Johnson and Odell [JO] on X_p.

In Section 1 we recall some standard definitions, state our basic conditions and prove the main result on constructing UFDD's. The main methods of

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proof are the blocking techniques and gliding hump arguments as used by various authors, e.g., [JO], [JZ]. The proof is rather technical so we have provided a sketch of the argument. (After reading this sketch the reader can proceed directly to the applications in Sections 2 and 3.) In order to apply our result we need to extend the results of [JRZ] on constructing bases in $\mathscr{L}_p$-spaces to construct bases with additional properties. This is done in Section 2. In Section 3 we apply the results to l_p and X_p.

Our notation for the most part follows standard Banach space theory conventions and unexplained items may be found in the books of Lindenstrauss and Tzafriri [LT], [LT-I, II] or Diestel [D].

1. Complemented subspaces of spaces with a UFDD

We first define a property which is critical to the proof of our main result.

Suppose that X is a Banach space with an FDD, $(E_n)_{n=1}^{\infty}$, and for each n let R_n be the FDD projection onto $[E_k : k \leqq n]$. We will say that a class of bounded linear operators $\mathcal{O}$ on X is *disjointly summable with respect to* $(E_n)_{n=1}^{\infty}$ if there is a constant $D < \infty$ such that given any increasing sequence of positive integers $k_1 < k_2 < \cdots$ and any sequence $(T_i)_{i=1}^{\infty}$ contained on $\mathcal{O}$, the operator

$$T = \sum_{i=1}^{\infty} (R_{k_i} - R_{k_{i-1}}) T_i (R_{k_i} - R_{k_{i-1}})$$

is bounded with $\| T \| \leqq D \sup \| T_i \|$.

Let us note that for the spaces l_p and c_0, $\mathcal{O}$ may be taken to be all bounded operators on the space with the usual unit vector basis. In the case of Tsirelson space, $\mathcal{O}$ can also be taken to be all bounded operators, but one must choose an appropriate blocking of the basis, [CS, Theorem III.5]. For X_p considered as a subspace of L_p we can take the simultaneously L_2 and L_p continuous operators, [JO].

Also observe that although we have not required that the FDD be unconditional, for most useful classes $\mathcal{O}$ this will follow immediately from the definition.

We can now state our main theorem.

THEOREM 1.1. *Suppose that Y is a Banach space with a shrinking UFDD (Y_i) and $\mathcal{O}$ is a class of operators on Y which are disjointly summable with respect to (Y_i). Suppose that P is a projection from Y onto a subspace X with a shrinking FDD (X_n) and that the FDD projections, when composed with P, are* *in $\mathcal{O}$. Then X is isomorphic to $W_1 \oplus W_2$ where for each i, W_i has a UFDD (Z_{ij}) with each Z_{ij} isomorphic to a block of the FDD (X_n) and the UFDD projections are perturbations of restrictions of UFDD projections in Y. Moreover, if there is a constant K such that for any k, l, $X_{l,k} = [X_n : l \leqq n \leqq k]$ has a K unconditional basis with basis projections composed with the FDD projection onto $X_{n,k}$ in $\mathcal{O}$, then X has an unconditional basis.*

Before we begin the proof, we will outline the steps. Let (Q_n) be the FDD projections in X, i.e., $Q_n \Sigma x_i = x_n$ where $x_i \in X_i$ for each i and let (R_n) be the projections for the UFDD in Y. We will block the FDD in X so that the new FDD, $Z_i = [X_n : n \in A_i]$, $i \in \mathbf{N}$, sits nicely with respect to the UFDD for Y. By this we mean that the blocks will be chosen to be so long that the supports of the even blocks relative to the UFDD are essentially disjoint. Thus we will automatically have a UFDD for the even blocks. Let I_i be the essential support of Z_i so that for each i, $I_i \cap I_{i+2} = \varnothing$ and $\Sigma_{j \in I_i} R_j z \approx z$ for $z \in Z_i$. Define

$$S = \sum_{i=1}^{\infty} \sum_{n \in A_{2i}} Q_n \sum_{j \in I_i} R_j.$$

The disjoint summability condition will guarantee that S is bounded, but the key point is that if we make the blocks sufficiently long we will also get a UFDD for the images of the odd blocks under $I - S$. Figures 1 and 2 illustrate the idea. In Fig. 1 the supports of the original FDD subspaces are shown relative to

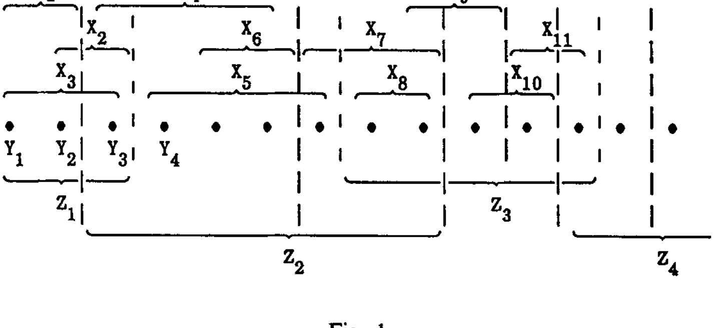

Fig. 1.

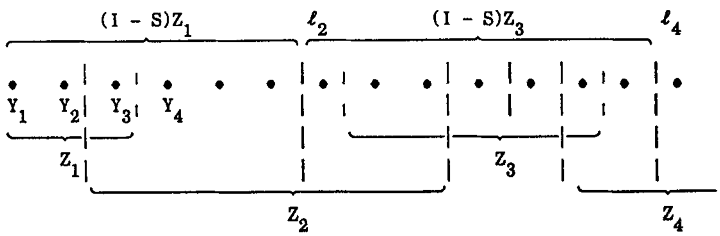

Fig. 2.

the UFDD for Y and the supports of the blocking into a new FDD are shown. Here $Z_1 = X_1 + X_2 + X_3$, $Z_2 = X_4 + \cdots + X_7$, and $Z_3 = X_8 + \cdots + X_{11}$. In Fig. 2 the supports of the images under $I - S$ of the odd blocks are shown. The l_i's mark the supports of these images.

Proof. For any integers $n < k$ let $R_{n,k} = \Sigma_{j=n}^{k-1} R_j$ and $Q_{n,k} = \Sigma_{j=n}^{k-1} Q_j$. Let $\varepsilon_k \downarrow 0$ such that $\Sigma\, \varepsilon_k < \varepsilon$, where ε will be determined later. Let C be $\| P \|$ times the FDD constant of (X_n) and B be the (suppression) UFDD constant for (Y_n). We will construct a blocking $Z_i = \Sigma_{n=n_i}^{n_{i+1}-1} X_n$, $i = 1, 2, \ldots$ and increasing sequences of integers (s_i), (t_i), and (l_i) satisfying

$$(0) \qquad s_{i-2} \leqq l_{i-1} \leqq t_i < l_i < s_i \qquad (t_1 = l_1 = s_0 = t_2 = 0)$$

and

$$(1) \quad \| R_{l_i,\infty} x \| < \varepsilon_i \| x \| \qquad \text{for all } x \in [Z_j : j \leqq i-1] = [X_n : n < n_{i-1}],$$

$$(2) \quad \| R_{1,l_i} x \| < \varepsilon_i \| x \| \qquad \text{for all } x \in [Z_j : i+1 \leqq j] = [X_n : n_{i+1} \leqq n],$$

$$(3) \qquad \| R_{l_i,\infty} Q_{1,n_{i+1}} P y \| < \varepsilon_i \| y \| \qquad \text{for all } y \in [Y_j : j \leqq s_{i-1}],$$

$$(4) \qquad \| R_{1,l_i} Q_{n_i,\infty} P y \| < \varepsilon_i \| y \| \qquad \text{for all } y \in [Y_j : t_{i+1} \leqq j],$$

$$(5) \qquad \| x - R_{t_i,s_i} x \| < \frac{\varepsilon_i}{4CB} \| x \| \qquad \text{for all } x \in Z_i.$$

(In the outline given above $A_i = \{n : n_i \leqq n < n_{i+1}\}$ and $I_i = \{j : t_i \leqq j < s_i\}$.)

Let us note that (5) implies

$$(6) \qquad \| x - R_{t_i,s_i} Q_{n_i,n_{i+1}} P R_{t_i,s_i} x \| < \varepsilon_i \| x \| \qquad \text{for all } x \in Z_i.$$

Indeed,

$$\| x - Q_{n_i,n_{i+1}} P R_{t_i,s_i} x \| = \| Q_{n_i,n_{i+1}} P (x - R_{t_i,s_i} x) \| \leqq \frac{\varepsilon_i 2C}{4CB} \| x \|$$

and thus

$$\| x - R_{t_i,s_i} Q_{n_i,n_{i+1}} P R_{t_i,s_i} x \| \leqq \| x - R_{t_i,s_i} x \| + \| R_{t_i,s_i} \| \, \| x - Q_{n_i,n_{i+1}} P R_{t_i,s_i} x \|$$

$$\leqq \frac{\varepsilon_i}{4CB} \| x \| + B \frac{\varepsilon_i 2C}{4CB} \| x \| < \varepsilon_i \| x \|$$

which is (6).

Roughly speaking $[t_i, s_i)$ is the support of Z_i and $[l_{i-1}, l_{i+1})$ is the support of the image of Z_i under certain restriction-projection operators. We will first prove the theorem assuming the above construction has been made and then describe the inductive construction of the blocking and sequences.

Let $\mathcal{R}_i = R_{t_i,s_i}$ and $\mathcal{Q}_i = Q_{n_i,n_{i+1}}$ for $i = 1, 2, \dots$. Consider the even blocks (Z_{2i}) and define an operator on Y by

$$T = \sum_{i=1}^{\infty} T_i \qquad \text{where } T_i = \mathcal{R}_{2i} \mathcal{Q}_{2i} P \mathcal{R}_{2i}.$$

The series converges by the disjoint summability property and we will show that T restricted to the span of the even blocks is an isomorphism.

If $x_{2i} \in Z_{2i}$ for each i then

$$\left\| \sum_{j=1}^{m} x_{2j} - T \sum_{j=1}^{m} x_{2j} \right\|$$

$$(7) \quad \leqq \left\| \sum_{j=1}^{m} [x_{2j} - T_{2j} x_{2j}] \right\| + \left\| \sum_{k=1}^{\infty} \sum_{j=k+1}^{m} T_{2k} x_{2j} \right\| + \left\| \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} T_{2k} x_{2j} \right\|$$

$$\leqq \sum_{j=1}^{m} \varepsilon_{2j} \| x_{2j} \| + \sum_{k=1}^{\infty} \left\| T_{2k} \sum_{j=k+1}^{m} x_{2j} \right\| + \sum_{j=1}^{\infty} \left\| T_{2k} \sum_{j=1}^{k-1} x_{2j} \right\| .$$

We will need to work a little to estimate the second and third terms. Note that $\Sigma_{j=1}^{k-1} x_{2j} \in [Z_j : j \leqq 2k-2]$ and so by (1)

$$\left\| T_{2k} \sum_{j=1}^{k-1} x_{2j} \right\| \leqq \| R_{t_{2k},s_{2k}} \| \, \| Q_{n_{2k}+1,n_{2k+1}} P \| \left\| R_{t_{2k},s_{2k}} \sum_{j=1}^{k-1} x_{2j} \right\|$$

$$\leqq B2CB \left\| R_{l_{2k-1},\infty} \sum_{j=1}^{k-1} x_{2j} \right\| \qquad \text{(because } l_{2k-1} \leqq t_{2k} \text{ in (0))}$$

$$\leqq 2B^2C\varepsilon_{2k-1} \left\| \sum_{j=1}^{k-1} x_{2j} \right\| \leqq 2B^2C^2\varepsilon_{2k-1} \left\| \sum_{j=1}^{m} x_{2j} \right\| .$$

It follows that the third term in (7) is at most

$$2B^2C^2 \sum_{k=1}^{\infty} \varepsilon_{2k-1} \left\| \sum_{j=1}^{m} x_{2j} \right\| .$$

Using (2) and observing that $s_{2k} \leqq l_{2k+1}$ in (0) and $\sum_{j=k+1}^{m} x_{2j} \in [Z_j : 2k+2 \leqq j]$, an analogous argument shows that

$$\left\| T_{2k} \sum_{j=k+1}^{m} x_{2j} \right\| \leqq 2B^2C\varepsilon_{2k+1} \left\| \sum_{j=k+1}^{m} x_{2j} \right\| \leqq 4B^2C^2\varepsilon_{2k+1} \left\| \sum_{j=1}^{m} x_{2j} \right\| .$$

Thus the second term in (7) is at most

$$4B^2C^2 \sum_{k=1}^{\infty} \varepsilon_{2k+1} \left\| \sum_{j=1}^{m} x_{2j} \right\| .$$

Combining the estimates we have proved that

$$\left\| (I-T) \sum_{j=1}^{m} x_{2j} \right\| \leqq [2C\varepsilon + 4B^2C^2\varepsilon + 2B^2C^2\varepsilon] \left\| \sum_{j=1}^{m} x_{2j} \right\|$$

$$\leqq 8B^2C^2\varepsilon \left\| \sum_{j=1}^{m} x_{2j} \right\| .$$

Therefore if $8B^2C^2\varepsilon < 1$, T restricted to $W_1 = [Z_{2j} : j \in \mathbf{N}]$, the span of the even blocks, is an isomorphism onto TW_1.

For technical reasons we will actually need to use a slight variant of T. For each i let $S_i = \mathcal{Q}_{2i} P \mathcal{R}_{2i}$ and $S = \sum S_i$. Because

$$\sum \| \mathcal{R}_{2i} x_{2i} - x_{2i} \| < \sum \varepsilon_{2i} \| x_{2i} \| / 4BC,$$

$$\| T - S \| \leqq \sum \varepsilon_{2i} \| \mathcal{Q}_{2i} P \mathcal{R}_{2i} \| / 4BC \leqq \varepsilon / 2.$$

Thus S is bounded and, if ε is small enough, S is an isomorphism from W_1 onto W_1 and $\tilde{P} = (S\,|_{W_1})^{-1} S$ is a projection onto W_1. Clearly W_1 has a UFDD.

Now suppose that there is a constant $K < \infty$ and, for each j, P_j is an operator on Z_{2j}, such that $P_j \mathcal{Q}_{2j} P \in \mathcal{O}$ and $\| P_j \| \leqq K$. Then $\Sigma\, \mathcal{R}_{2j} P_j \mathcal{Q}_{2j} P \mathcal{R}_{2j}$ is bounded on Y and it follows that $\Sigma\, P_j \mathcal{Q}_{2j}$ is bounded on W_1. Therefore if each block Z_{2j} has an unconditional basis with projections which when composed with $\mathcal{Q}_{2j} P$ are in $\mathcal{O}$ then we get that the span of the even blocks has an unconditional basis.

Next we need to consider the kernel of the projection $\tilde{P}$. We would be done if the kernel were exactly the span of the odd blocks, but this is not necessarily the case. We will show that the kernel is the closed span of the images of these odd blocks and the images are essentially disjointly supported. First we observe that

$$(I - \tilde{P}) \big|_{Z_{2j+1}} \text{ is an isomorphism for each } j. \tag{8}$$

Indeed, because $\Sigma_k Z_k$ is an FDD there is a constant A such that

$$\| x - x_{2j+1} \| \geqq A \| x_{2j+1} \| \quad \text{for all } x \in [Z_k : k \neq 2j+1] \text{ and } x_{2j+1} \in Z_{2j+1}.$$

But $\operatorname{range}(\tilde{P}) \subset [Z_{2k} : k \in \mathbf{N}] \subset [Z_k : k \neq 2j+1]$, thus $\| (I - \tilde{P}) x_{2j+1} \| \geqq A \| x_{2j+1} \|$.

The main chore remaining is to show

CLAIM. There is a constant K_1 which does not depend on (ε_j) such that

$$\| R_{l_{2j}, l_{2j+2}} z - z \| \leqq \varepsilon_{2j-1} K_1 \| z \| \qquad \text{for all } z \in (I - \tilde{P}) Z_{2j+1}.$$

Once this is proved we can immediately conclude that the kernel of $\tilde{P}$ has a UFDD.

The proof of the claim requires a number of estimates. First note that

$$\| (I - \tilde{P}) x \| = \| x - (S \big|_{W_1})^{-1} S x \| \leqq \| x - Sx \| + \| Sx - (S \big|_{W_1})^{-1} S x \|.$$

We will see below that $(S \big|_{W_1})^{-1}$ is a perturbation of the identity on W_1. Thus we will concentrate on the first term and prove

CLAIM′. There is a constant K_2 which does not depend on (ε_j) such that

$$\| R_{l_{2j}, l_{2j+2}} z - z \| \leqq \varepsilon_{2j-1} K_2 \| z \| \qquad \text{for all } z \in (I - S) Z_{2j+1}.$$

First observe that if $x \in Z_{2j+1}$, $l_{2j} \leqq t_{2j+1} < s_{2j+1} \leqq l_{2j+2}$ and thus

$$\| x - R_{l_{2j}, l_{2j+2}} x \| < 2B\varepsilon_{2j+1} \| x \| / 4CB \tag{9}$$

by (5). Therefore we need to estimate $\| R_{1, l_{2j}} Sx \|$ and $\| R_{l_{2j+2}, \infty} Sx \|$. To do this we will break Sx into three parts, namely,

$$\sum_{k=1}^{j-1} S_k x, \quad S_j x + S_{j+1} x, \quad \text{and} \quad \sum_{k=j+2}^{\infty} S_k x.$$

Because $S_k x = S_k R_{l_{2j+2},\infty} x$ for $k \geqq j+2$ and $S_k x = S_k R_{1,l_{2j}} x$ for $k \leqq j-1$, it follows from (1) and (2) that

$$\left\| \sum_{k=j+2}^{\infty} S_k x \right\| \leqq \varepsilon_{2j+2}(D2C + \varepsilon/2) \| x \| \tag{10}$$

and

$$\left\| \sum_{k=1}^{j-1} S_k x \right\| \leqq \varepsilon_{2j}(D2C + \varepsilon/2) \| x \| . \tag{11}$$

Also from (5) we have that $\| x - R_{t_{2j+1},\infty} x \| < \varepsilon_{2j+1} \| x \| /2C$. Hence

$$\begin{aligned} \| R_{1,l_{2j}} S_j x \| &\leqq \| R_{1,l_{2j}} S_j R_{t_{2j+1},\infty} x \| + \| R_{1,l_{2j}} S_j (x - R_{t_{2j+1},\infty} x) \| \\ &\leqq \| R_{1,l_{2j}} Q_{n_{2j},\infty} P R_{t_{2j},s_{2j}} R_{t_{2j+1},\infty} x \| \\ &\qquad + \| R_{1,l_{2j}} Q_{n_{2j+1},\infty} P R_{t_{2j},s_{2j}} R_{t_{2j+1},\infty} x \| + \| R_{1,l_{2j}} S_j \| \varepsilon_{2j+1} \| x \| /2C \\ &< \varepsilon_{2j}(\| R_{t_{2j+1},s_{2j}} x \| + \| Q_{n_{2j+1},\infty} P R_{t_{2j+1},s_{2j}} x \|) + B^2 C \varepsilon_{2j+1} \| x \| /2C \\ &\qquad\qquad \text{(by (4) and (2))} \\ &< [\varepsilon_{2j}(B + CB) + B^2 \varepsilon_{2j+1}/2] \| x \| < 3B^2 C \varepsilon_{2j} \| x \| . \end{aligned}$$

Because $R_{1,l_{2j}} S_k x \in Z_{2k}$ for $k > j$, it follows from (2) and the fact that the S_k's are close to the T_k's that

$$\left\| R_{1,l_{2j}} \sum_{k=j+1}^{\infty} S_k x \right\| < \varepsilon_{2j} \left\| \sum_{k=j+1}^{\infty} S_k x \right\| \leqq \varepsilon_{2j}(2CD + \varepsilon/2) \| x \| . \tag{12}$$

Combining (11) and (12) with the estimate on $R_{1,l_{2j}} S_j x$ we get that

$$\left\| R_{1,l_{2j}} \sum_{k=1}^{\infty} S_k x \right\| < 4(B + D) B C \varepsilon_{2j} \| x \| . \tag{13}$$

Again from (5) we have that $\| x - R_{1,s_{2j+1}} x \| < \varepsilon_{2j+1} \| x \| /2C$. Hence

$$\begin{aligned} &\| R_{l_{2j+2},\infty} S_{j+1} x \| \\ &\quad \leqq \| R_{l_{2j+2},\infty} S_{j+1} R_{1,s_{2j+1}} x \| + \| R_{l_{2j+2},\infty} S_{j+1} (x - R_{1,s_{2j+1}} x) \| \\ &\quad \leqq \| R_{l_{2j+2},\infty} Q_{1,n_{2j+3}} P R_{t_{2j+2},s_{2j+2}} R_{1,s_{2j+1}} x \| \\ &\qquad + \| R_{l_{2j+2},\infty} Q_{1,n_{2j+2}} P R_{t_{2j+2},s_{2j+2}} R_{1,s_{2j+1}} x \| + \| R_{l_{2j+2},\infty} S_j \| \varepsilon_{2j+1} \| x \| /2C \end{aligned}$$

$$< \varepsilon_{2j+2}(\| R_{t_{2j+2},s_{2j+1}}x \| + \| Q_{1,n_{2j+2}}PR_{t_{2j+2},s_{2j+1}}x \|) + B^2C\varepsilon_{2j+1}\| x \| /2C \qquad \text{(by (3) and (1))}$$

$$< [\varepsilon_{2j+2}(B + CB) + B^2\varepsilon_{2j+1}/2] \| x \| < 3B^2C\varepsilon_{2j+1}\| x \| .$$

Because $S_kx \in Z_{2k}$ for $k < j+1$, it follows from (1) that

$$\left\| R_{l_{2j+2},\infty} \sum_{k=1}^{j} S_kx \right\| < (2DC + \varepsilon/2)\varepsilon_{2j+2}\| x \| . \tag{14}$$

Combining (10) and (14) with the estimate for $\| R_{l_{2j+2}}S_{j+1}x \|$ we get that

$$\left\| R_{l_{2j+2},\infty} \sum_{k=1}^{\infty} S_kx \right\| < 4(D+B)BC\varepsilon_{2j+1}\| x \| . \tag{15}$$

Claim′ follows from (9), (13), and (15). To get the Claim we must estimate $\| Sx - \tilde{P}x \|$. Because $Sx \in W_1$, $Sx - \tilde{P}x = \Sigma_{n=0}^{\infty} (I - S)^n(S - S^2)x$. Therefore it is enough to estimate $\| (S - S^2)x \|$:

$$\begin{aligned}
&\| Sx - S^2x \| \\
&\leqq \| I - S \| \left\| \sum_{n \neq j,j+1} S_nx \right\| + \left\| \sum_{n \neq j,j+1} S_n(S_j + S_{j+1})x \right\| \\
&\quad + \| (S_j + S_{j+1})x - (S_j + S_{j+1})^2x \| \\
&\leqq \| I - S \| \, 4DC\varepsilon_{2j} \| x \| \\
&\quad + \left[\left\| \sum_{n > j+1} S_n \right\| \varepsilon_{2j+3} + \left\| \sum_{n<j} S_n \right\| \varepsilon_{2j-1} \right] \| (S_j + S_{j+1})x \| \\
&\quad + \| S_jx - S_j^2x \| + \| S_{j+1}x - S_{j+1}^2x \| + \| S_jS_{j+1}x \| + \| S_{j+1}S_jx \| \\
&\qquad \text{(by (10) and (11) and by (1) and (2))} \\
&\leqq K_2\varepsilon_{2j-1}\| x \| + \frac{\varepsilon_{2j} + \varepsilon_{2j+2}}{2B} \| x \| \\
&\quad + \| S_j \| \varepsilon_{2j+1} \| x \| + \| S_{j+1} \| \varepsilon_{2j+1} \| x \|
\end{aligned}$$

where K_2 is some constant which does not depend on j and the other estimates follow from the proof of (6) and from (1) and (2). This establishes the Claim.

The Claim implies that $W_2 = \ker \tilde{P} = [(I - \tilde{P})Z_{2j+1} : j \in \mathbf{N}]$ is isomorphic to $[R_{l_{2j},l_{2j+2}}(I - \tilde{P})Z_{2j+1} : j \in \mathbf{N}]$ which clearly has a UFDD. In particular note that W_2 is an unconditional sum of isomorphs of the blocks $Z_{2j+1}, j \in \mathbf{N}$. (Note that we are *not* claiming that $(I - \tilde{P})$ is an isomorphism on $[Z_{2j+1} : j \in \mathbf{N}]$.)

Finally we need to observe that $\mathcal{Q}_{2j+1}|_{(I-\tilde{P})Z_{2j+1}}$ is the inverse of $(I-\tilde{P})|_{Z_{2j+1}}$. Thus if for each j, P_j is a bounded operator on Z_{2j+1} and $P_j\mathcal{Q}_{2j+1}P \in \mathcal{O}$, $\Sigma_{j=1}^{\infty} R_{l_{2j},l_{2j+2}}P_j\mathcal{Q}_{2j+1}PR_{l_{2j},l_{2j+2}}$ is bounded on Y by the disjoint summability condition. The Claim and (9) imply that $M = \Sigma_{j=1}^{\infty} P_j\mathcal{Q}_{2j+1}P$ is bounded from W_2 into $[Z_{2j+1} : j \in \mathbf{N}]$. Therefore $(I-\tilde{P})M$ is bounded on W_2. This immediately gives the unconditional basis for W_2 if each block has an unconditional basis with projections in $\mathcal{O}$.

Now we will describe the inductive construction of the blocking and the sequences of integers (t_i), (s_i), and (l_i). Let $n_1 = 1$ and let n_2 be any larger integer. Clearly there is an integer s_1 such that (5) is satisfied for $Z_1 = [X_n : n_1 \leqq n < n_2]$. Taking $i = 2$ and using the finite dimensionality of Z_1 and $[Y_j : j < s_1]$ find l_2 satisfying (1) and (3) for any n_3. Next we choose $t_3 > l_2$ satisfying (4) for any n_3 and $i = 2$ by using the shrinking property of the FDD's. Finally choose n_3 such that for all $x \in [X_n : n \geqq n_3]$, $\| R_{1,t_3}x \| < \varepsilon_3/8BC$. This completes the first cycle of the construction.

Suppose we have chosen n_j, s_{j-2}, l_{j-1}, and t_j satisfying (0)–(5) for $j \leqq k$ and $\| R_{1,t_j}x \| < \varepsilon_j/8BC, j = k-1, k$. Find $s_{k-1} > l_{k-1}$ satisfying (5) for $i = k-1$. Next choose $l_k > \max(t_k, s_{k-1})$ satisfying (1) and (3) for any n_{k+1} by using the finite dimensionality of $[Y_j : j \leqq s_{k-1}]$. Using the shrinking assumption choose $t_{k+1} \geqq \max(s_{k-1}, l_k)$ satisfying (4). To complete the inductive step choose n_{k+1} satisfying

$$\| R_{1,t_{k+1}}x \| < \frac{\varepsilon_{k+1}}{8BC} \qquad \text{for } x \in [X_n : n \geqq n_{k+1}].$$

This shows that the required blocking and sequences satisfying (0)–(5) can be found and completes the proof of the theorem. Q.E.D.

2. Constructing bases in complemented subspaces of X_p

In this section we will modify the idea presented in [JRZ] to construct special bases in complemented subspaces of X_p. In order to state precisely our results we need to introduce some special notation. Throughout this section $w = (w_n)$ will be a decreasing sequence of positive real numbers and $2 < p < \infty$.

For each n, $k \in \mathbf{N}$ let $X_{p,w}^{n,k} = \{(a_{i,j}) : 1 \leqq i \leqq n,\ 1 \leqq j \leqq k\} = \mathbf{R}^{nk}$ with the norm

$$\| (a_{i,j}) \|_{n,k,w} = \max\{ |(a_{i,j})|_p, |(a_{i,j})|_{2,w} \}$$

where

$$|(a_{i,j})|_p = \left(\sum_{i,j} |a_{i,j}|^p\right)^{1/p} \quad \text{and} \quad |(a_{i,j})|_{2,w} = \left(\sum_i \sum_j |a_{i,j}|^2 w_i^2\right)^{1/2}.$$

The Rosenthal space X_p is $X_{p,w}^{\infty,1}$ where $w_i \downarrow 0$ and $\Sigma\, w_i^{2p/(p-2)} = \infty$ and as usual we will write $X_{p,w}$ for $X_{p,w}^{\infty,1}$. If (X_i) is a sequence of spaces such that for each i, $\|\cdot\|_i = \max\{|\cdot|_{i,p}, |\cdot|_{i,2}\}$, where $|\cdot|_{i,p}$ and $|\cdot|_{i,2}$ are norms on X_i, we define the formal p, 2-sum $(\Sigma \oplus X_i)_{p,2}$ as the completion of the space of sequences (x_i), $x_i \in X_i$ for each i and $x_i \neq 0$ for only finitely many i, with

$$\|(x_i)\|_{p,2} = \max\left\{\left(\sum_i |x_i|_{i,p}^p\right)^{1/p}, \left(\sum_i |x_i|_{i,2}^2\right)^{1/2}\right\}.$$

Note that $(\Sigma_{i=1}^{\infty} \oplus X_{p,w}^{n_i,k_i})_{p,2}$ is isomorphic to the Rosenthal space X_p if and only if n_i goes to ∞, $w_i \downarrow 0$, and $\Sigma_{i=1}^{\infty} \Sigma_{j=1}^{n_i} k_i w_j^{2p/(p-2)} = \infty$, [R].

It will be necessary to work with both the p-norm and 2-norm on X_p. In order to make this a little easier we will introduce the notion of p, 2-bounded operator. Suppose X_1 and X_2 are Banach spaces with norm $\|\cdot\|_i = \max\{|\cdot|_{i,p}, |\cdot|_{i,2}\}$ for $i = 1, 2$. We will say that an operator T from X_1 to X_2 is C p, 2-bounded if C is a constant such that $|Tx|_{2,p} \leqq C|x|_{1,p}$ and $|Tx|_{2,2} \leqq C|x|_{1,2}$ for all $x \in X_1$. We will write $\|T\|_{p,2}$ for the smallest such C. For example, if X_1 and X_2 are subspaces of $L_p[0,1]$ we can let the p-norm be the L_p norm and the 2-norm be the $L_2[0,1]$ norm. Thus any operator from $L_2[0,1]$ to itself which is bounded on $L_p[0,1]$ as well, and maps X_1 into X_2, is p, 2-bounded. We will, in particular, be interested in complemented subspaces of X_p. We will say that a subspace X of a space Y (with norm as above) is p, 2-complemented if there is a p, 2-bounded idempotent operator on Y with range equal to X.

Our interest in p, 2-bounded operators stems from the following observation which was employed in the proof that X_p is primary [JO]. Suppose that $2 < p < \infty$ and that $X_{p,w}$ is isomorphically embedded in L_p such that the L_2 norm is equivalent to $|\cdot|_2$ part of the $X_{p,w}$ norm, i.e., if (x_i) is equivalent to the standard basis of $X_{p,w}$, $\|\Sigma\, a_i x_i\|_2 \sim (\Sigma\, a_i^2 w_i^2)^{1/2}$. If (T_j) is a sequence of uniformly p, 2-bounded operators on $[x_i : i \in \mathbf{N}]$, (k_j) is a strictly increasing sequence of integers and $x \in [x_i : i \in \mathbf{N}]$,

$$\left\|\sum_{j=1}^{\infty} R_{k_j,k_{j+1}} T_j R_{k_j,k_{j+1}} x\right\|_p$$

$$\sim \max\left\{\left(\sum_{j=1}^{\infty} \sum_{i=k_j}^{k_{j+1}-1} [(T_j R_{k_j,k_{j+1}} x)(i)]^p\right)^{1/p}, \left(\sum_{j=1}^{\infty} \sum_{i=k_j}^{k_{j+1}-1} w_i^2 [(T_j R_{k_j,k_{j+1}} x)(i)]^2\right)^{1/2}\right\}$$

$$\leqq C_1 \max\left\{\left(\sum_{j=1}^{\infty}[\| T_j \| \; \| R_{k_j,k_{j+1}}x \|_p]^p\right)^{1/p}, \left(\sum_{j=1}^{\infty}[\| T_j \|_2 \| R_{k_j,k_{j+1}}x \|_2]^2\right)^{1/2}\right\}$$

$$\leqq C_2 \sup_j \| T_j \|_{p,2} \max\left\{\left(\sum_{j=1}^{\infty}\sum_{i=k_j}^{k_{j+1}-1}[x(i)]^p\right)^{1/p}\right.$$

$$\left. + \left(\sum_{j=1}^{\infty}\left(\sum_{i=k_j}^{k_{j+1}-1}[x(i)]^2 w_i^2\right)^{p/2}\right)^{1/p}, \| x \|_2\right\}.$$

(C_1 and C_2 are constants from the equivalence of the $X_{p,w}$ and L_p norms.) Because $p > 2$,

$$\left(\sum_{j=1}^{\infty}\left(\sum_{i=k_j}^{k_{j+1}-1}[x(i)]^2 w_i^2\right)^{p/2}\right)^{1/p} \leqq \left(\sum_{j=1}^{\infty}\sum_{i=k_j}^{k_{j+1}-1}[x(i)]^2 w_i^2\right)^{1/2} \sim \| x \|_2.$$

Thus $\| \Sigma_{j=1}^{\infty} R_{k_j,k_{j+1}} T_j R_{k_j,k_{j+1}} \|$ is bounded by a constant times $\sup_j \| T_j \|_{p,2}$, i.e., such operators satisfy the disjoint summability condition.

In Section 3 we will use this observation to prove the following result.

THEOREM 2.1. *Let $w = (w_n)$ be a decreasing sequence of positive real numbers with limit 0 such that $\Sigma\, w_i^{2p/(p-2)} = \infty$. Suppose that X is a $p, 2$-complemented subspace of $X_{p,w}$. Then the following are equivalent*:

(i) *There is a constant K such that for every n there exists a K $p, 2$-complemented subspace X_n of X which is K $p, 2$-isomorphic to $X_{p,w}^{n,1}$,*

(ii) *X is $p, 2$-isomorphic to $X_{p,w}$.*

The proof will use the result from Section 1. Consequently we need to build special UFDD's in complemented subspaces of X_p. In [JRZ] bases for $\mathscr{L}_p$ spaces were constructed and our argument mimics that argument in the form presented in [LT, p. 203]. The basic difficulty is that we must carry out the construction with two norms on the space. In the case of X_p these norms will be the L_2 and L_p norms. Our first lemma is a multi-norm version of the principle used to prove [LT, Proposition II.5.10].

LEMMA 2.2. *Suppose that $\| \cdot \|_1, \| \cdot \|_2, \ldots, \| \cdot \|_k$ are norms on a space Z (not necessarily closed under each) and for each i let Z_i be the completion of Z with respect to $\| \cdot \|_i$. Let P_n be a sequence of projections on Z such that $\lim_{n\to\infty} \| P_n z - z \|_i = 0$ for all $z \in Z$ and $i = 1, 2, \ldots, k$ and such that there are constants $C_{i,n}$ for which*

$$C_{i,n}^{-1} \| z \|_i \leqq \| z \|_1 \leqq C_{i,n} \| z \|_i$$

for all $z \in \text{range}(P_n) \cap Z_1 \cap Z_i$. *Finally let* F *be a finite dimensional subspace of* Z *and suppose that* (Q_n) *is a sequence of uniformly bounded operators on* $(Z, \|\cdot\|_i)$ *such that* $\lim_{m\to\infty} \| Q_m x - x \|_i = 0$ *for all* $x \in F$ *and each* i. *Then for every* $\varepsilon > 0$ *there exists an operator* Q *such that* $\sup_i \| Q_m - Q \|_i < \varepsilon$ *for some* m *and* $Qx = x$ *for all* $x \in F$. *If* Q_m *is a projection*, Q *can also be chosen to be a projection*. *Moreover* $\text{range}(Q) \subset \text{range}(Q_m) + F$.

PROOF. Let δ_1 and δ_2 be positive numbers. Choose n such that $\| P_n x - x \|_i < \delta_1 \| x \|_i$ for all $x \in F$ and $i = 1, 2, \dots, k$, and choose l such that $\| Q_l x - x \|_i < \delta_2 \| x \|_i$ for all $x \in F$ and $i = 1, 2, \dots, k$. Observe that $P_n(F)$ is complemented in range(P_n) with a projection of norm at most $\dim F \cdot \max_i C_{i,n}^2$. Indeed, find a projection R from range(P_n) onto $P_n(F)$ with $\| R \|_1 \leqq \dim F$. Then

$$\| Rz \|_i \leqq C_{i,n} \| Rz \|_1 \leqq C_{i,n} \dim F \| z \|_1 \leqq C_{i,n}^2 \dim F \| z \|_i.$$

Thus $P_n(F)$ is complemented in Z with projection RP_n and

$$\| RP_n \|_i \leqq C_{i,n}^2 \dim F \| P_n \|_i \qquad \text{for } i = 1, 2, \dots, k.$$

Next we will show that $Q_l(F)$ is complemented in range(Q_l). Note that

$$\| P_n x - Q_l x \|_i \leqq \| P_n x - x \|_i + \| x - Q_l x \|_i < (\delta_1 + \delta_2) \| x \|_i$$

and thus if $x \in F$,

$$\begin{aligned} \| RP_n Q_l x - Q_l x \|_i &\leqq \| RP_n Q_l x - RP_n x \|_i + \| P_n x - Q_l x \|_i \\ &< \| RP_n \|_i \| Q_l x - x \|_i + (\delta_1 + \delta_2) \| x \|_i \\ &< (\| RP_n \|_i \delta_2 + \delta_1 + \delta_2) \| Q_l x \|_i (1 - \delta_2)^{-1}. \end{aligned}$$

Therefore if δ_1 and δ_2 are small enough, RP_n is an isomorphism from $Q_l(F)$ onto $P_n(F)$ and $P = (RP_n |_{Q_l(F)})^{-1} RP_n Q_l$ is a projection onto $Q_l(F)$. Moreover,

$$\begin{aligned} \| RP_n Q_l x \|_i &\geqq \| Q_l x \|_i \left[1 - \frac{(\delta_1 + \delta_2) + \| RP_n \|_i \delta_2}{1 - \delta_2} \right] \\ &\geqq \| Q_l x \|_i \left[1 - \frac{(\delta_1 + \delta_2) + C_{i,n}^2 \dim F \| P_n \|_i \delta_2}{1 - \delta_2} \right]. \end{aligned}$$

Thus

$$\| P \|_i \leqq \left[1 - \frac{(\delta_1 + \delta_2) + C_{i,n}^2 \dim F \| P_n \|_i \delta_2}{1 - \delta_2} \right]^{-1} C_{i,n}^2 \dim F \| P_n \|_i \| Q_l \|_i.$$

To complete the proof let

$$Q = (PQ_l|_F)^{-1}PQ_l + (I - P)Q_l = Q_l + ((PQ_l|_F)^{-1} - I)PQ_l.$$

Then $\| Q - Q_l \|_i \leqq \| [(PQ_l|_F)^{-1} - I]|_{Q_l(F)} \|_i \| PQ_l \|_i$. Because we first choose n, we can choose δ_2 to make $C^2_{i,n} \| P_n \|_i \delta_2$ as small as we like. Therefore $\max \| Q - Q_l \|_i$ can be made less than ε, as required. Direct computation of Q^2 shows that Q is a projection if Q_l is. Q.E.D.

LEMMA 2.3. *For each l, n, and $k \in \mathbf{N}$, $\varepsilon > 0$, and $K < \infty$ there is an integer N such that if X is an l dimensional subspace of $X_{p,w}$ and $Y \subset X_{p,w}$ is $p,2$-isomorphic to $X^{n;N}_{p,w}$, with constant K and $p,2$-complemented with constant K in $X_{p,w}$, then there is a $p,2$-complemented subspace Z of Y which is $p,2$-isomorphic to $X^{n;k}_{p,w}$ (with constants depending only on K and ε) such that if Q is the projection from $X_{p,w}$ onto Z then $\| Q|_X \|_{p,2} < \varepsilon$.*

PROOF. Let $(x_i)^l_{i=1}$ be a normalized basis for X and let P be the given projection onto Y. Let $(y^*_{i,j})$ be the coordinate functionals for the standard $X^{n,N}_p$ basis of Y. For any $\delta > 0$ and N sufficiently large for each $i \leqq n$ there are k integers $j(i, m)$, $m = 1, 2, \ldots, k$ such that $|y^*_{i,j(i,m)}(Px_s)| < \delta$ for $1 \leqq s \leqq l$, $m = 1, 2, \ldots, k$. If δ is sufficiently small, it follows that

$$\left| \sum_{i=1}^{n} \sum_{m=1}^{k} y^*_{i,j(i,m)}(Px) y_{i,j(i,m)} \right|_r < \varepsilon |x|_r$$

for all $x \in X$ and $r = p$ or 2. Hence we may choose

$$Z = [y_{i,j(i,m)} : 1 \leqq i \leqq n,\ 1 \leqq m \leqq k]$$

and use the natural projection in Y. Q.E.D.

As stated Lemma 2.3 is without content in the cases for which $X_{p,w}$ is isomorphic to l_p or $l_p \oplus l_2$ because in those cases $X^{n,N}_{p,w}$ is not uniformly $p,2$-isomorphic to a subspace of $X_{p,w}$. Of course analogous results can be proved for those cases.

LEMMA 2.4. *Let F be a finite dimensional subspace of $X_{p,w}$ and $\varepsilon > 0$. Suppose that Q is a finite rank $p,2$-continuous projection on $X_{p,w}$ such that $\| Qf - f \|_{p,2} < \delta \| f \|_{p,2}$ for all $f \in F$. Then if $\delta < \delta(F, \| Q \|, \varepsilon)$ there is a finite rank $p,2$-continuous projection Q_1 on $X_{p,w}$ such that $Q_1 f = f$ for all $f \in F$ and $\| Q_1 - Q \|_{p,2} < \varepsilon$.*

PROOF. Let (P_n) be the basis projections for the standard basis of $X_{p,w}$

and suppose that there were no such $\delta(F, \| Q \|, \varepsilon)$. Then there would exist $\delta_l \downarrow 0$ and a uniformly bounded sequence (Q_l) of projections such that $\| Q_l f - f \|_{p,2} < \delta_l \| f \|_{p,2}$ for all $f \in F$, for which the conclusion of the lemma is false. This would contradict Lemma 2.2. Q.E.D.

We are now ready to prove our version of [LT, Proposition II.5.9].

PROPOSITION 2.5. *Suppose that X is a $p, 2$-complemented subspace of $X_{p,w}$ and $X_{p,w}^{n,n}$ is $p, 2$-isomorphic to a $p, 2$-complemented subspace X_n of X for all n with bounds independent of n. Then every finite dimensional subspace B of X is contained in a finite dimensional subspace F of X which is $p, 2$-isomorphic to $X_{p,w}^{n,1}$ for some n and $p, 2$-complemented in X. Moreover the constants depend only on the $p, 2$-norm of the projections onto X and the X_n's and the $p, 2$-distance of the X_n's to $X_{p,w}^{n,n}$.*

PROOF. Let $\delta > 0$. Let (P_n) be the basis projections for $X_{p,w}$ and choose n so large that B is essentially contained within the range of P_n. Find a projection P' such that $B \subset \text{range}(P')$ and $\| P_n - P' \|_{p,2} < \delta$. Let Q be the projection onto X. By Lemma 2.3 there is a $p, 2$-complemented subspace Z of X which is $p, 2$-isomorphic to $X_{p,w}^{n,1}$ and such that the projection R of X_p onto Z satisfies $\| R \,|_{\text{range}(QP')} \|_{p,2} < \delta / n$.

We need to perturb R so that $RQP' = 0$. To do this let Q' be a projection onto range(QP') with $\| Q' \|_{p,2}$ of order $n = \text{rank}(P')$. Then if δ is small enough $Q'' = (Q'(I-R)\,|_{\text{range}(QP')})^{-1} Q'(I-R)$ is a projection onto range(QP') such that ker $(Q'') \supset \text{range}(R)$. Therefore $R(I - Q'')$ is a $p, 2$-bounded projection onto Z with kernel containing range(Q'') = range(QP'). We will assume that R is the perturbed operator.

Let τ be the canonical isomorphism from range $W = \text{range}(P')$ onto Z and observe that τ is a $p, 2$ isomorphism. Define T from W into X by

$$T = Q \,|_W + \tau(I - P'Q) \,|_W.$$

Clearly $T \,|_B = I \,|_B$ and $\| T \|_{p,2} < \| Q \|_{p,2} + \| \tau \|_{p,2}(1 + \| Q \|_{p,2})$. If $x \in W$

$$\begin{aligned} \| Tx \|_{p,2} &= \| Qx + \tau(I - P'Q)x \|_{p,2} \\ &\geq (\| R \|_{p,2}^{-1} \| \tau(I - P'Q)x \|_{p,2} + (1 + \| R \|_{p,2})^{-1} \| Qx \|_{p,2})/2 \\ &\geq (1 + \| R \|_{p,2}^{-1})(\| \tau \|_{p,2} \| (I - P'Q)x \|_{p,2} + (1 + \delta)^{-1} \| P'Qx \|_{p,2})/2. \end{aligned}$$

Thus T is a $p, 2$-isomorphism on W which is the identity on B. Let $F = T(W)$. F is $p, 2$-isomorphic to $X_p^{n,1}$ and if $S = P'(I - R) + \tau^{-1}R$, $STx = x$ for all

$x \in W$. Hence TS is a p, 2-continuous projection onto F. Q.E.D.

Within the proof of Proposition 2.5 is a shifting argument which prevent this approach from working in the case of $l_p \oplus l_2$. The difficulty is actually in the l_p part. Indeed if we let $X = [1_{A_i} : i \in \mathbf{N}]$ where the A_i's are disjoint subsets of $[0, 1]$, then spaces $X_n = [1_{A_i} : n \leqq i]$ are not uniformly p, 2-isomorphic. Thus we cannot find the subspaces Z used in the proof by simply gliding out along the basis.

Our final result in this section shows us that we can construct p, 2-FDD's in "large" subspaces of X_p.

Proposition 2.6. *Suppose that X is a p, 2-complemented subspace of X_p and that (Q_n) is a sequence of p, 2-continuous finite rank projections on X such that* $\lim_n \| Q_n x - x \|_{p,2} = 0$ *for all $x \in X$. Then there is a sequence of projections (Q'_n) on X such that*

(i) *Q'_n is p, 2-continuous for each n,*

(ii) *$Q'_n Q'_m = Q'_{\min(n,m)}$ for all n and m,*

(iii) $\lim_n \| Q'_n x - x \| = 0$ *for all $x \in X$.*

Moreover, if range(Q_n) *is K p, 2-isomorphic to $X_p^{k(n)}$ ($k(n) =$* dim range(Q_n)) *for each n, then for any $\varepsilon > 0$ the sequence (Q'_n) may be chosen such that* range$(Q'_n - Q'_m)$ *is $K(1+\varepsilon)\,p$, 2-isomorphic to $X_p^{k(n,m)}$, where $k(n, m) =$* dim range$(Q'_n - Q'_m)$.

Proof. First observe that the basis projections (P_n) in X_p and (P_n^*) in X^* satisfy the hypothesis of Lemma 2.2 where the norms are the p and 2, respectively, q and 2, norms. We will use this to prove the following

Claim. For every $B \subset X$ and $C \subset X^*$, finite dimensional, there exists a p, 2-continuous finite rank projection Q in X with $QX \supset B$, $Q^*X^* \supset C$, and $\| Q \|_{p,2} \leqq f(\lambda)$, where $\lambda = \sup \| Q_n \|_{p,2}$, and range$(Q)$ is $(1+\varepsilon)\,p$, 2-isomorphic to range Q_k for some k.

Once this claim is proved the argument follows just as in [LT, p. 206].

To prove the claim, observe that because X is reflexive $Q_n^* x^* \rightarrow x^*$ for all x^* in $(X, |\cdot|_p)^*$ and in $(X, |\cdot|_2)^*$. Hence there are convex combinations T_k of the Q_n's, i.e., $T_k = \Sigma_{i=1}^m \lambda_i^k Q_i^*$ where $\lambda_i^k \geqq 0$ and $\Sigma_{i=1}^m \lambda_i^k = 1$, such that $T_k x^* \rightarrow x^*$ in the norm topology of $(X, |\cdot|_p)^*$ for all x^* in $(X, |\cdot|_p)^*$. Indeed, let (x_j^*) be a norm dense sequence in $(X, |\cdot|_p)^*$ and use Mazur's Theorem repeatedly to get such convex combinations for $x_j^*, j = 1, 2, \ldots, n$, for any n. A diagonalization argument produces the required T_k's.

If we repeat this argument using the T_k's and $(X, |\cdot|_2)^*$ we can find convex combinations of the T_k's (and therefore of the Q_i^*'s) to get a sequence of operators (S_k) on X^* such that for all $x^* \in Y = (X, |\cdot|_p)^* \cap (X, |\cdot|_2)^*$, $|S_k x^* - x^*|_{r^*} \to 0$ for $r = p$ or 2. ($|\cdot|_{r^*}$ denotes the dual norm.) Clearly Y is norm dense in X^*, so we have that $\| S_k x^* - x^* \|_{p,2} \to 0$ for all $x^* \in X^*$.

We may apply Lemma 2.2 with (P_n^*), C, and (S_k) to get a finite rank operator S on X^* such that $Sx^* = x^*$ for all $x^* \in C$ and $\| S \|_{p,2^*} \leqq 2 \sup \| Q_n \|_{p,2}$. Next apply Lemma 2.2 to (P_n), $S^*(X) + B$ and (Q_n) to get a finite rank projection P such that $P(X) \supset S^*(X) + B$ and $\| Q_n - P \|_{p,2} < \delta$. Let $Q = S^* + P - S^*P$. Q is the required projection. Also observe that range$(Q) \subset$ range(P) and if $Q_n x = x$, then

$$\begin{aligned}\| Q_n x - Qx \|_{p,2} &\leqq \| Q_n x - Px \|_{p,2} + \| S^*(I - P)x \|_{p,2} \\ &\leqq \delta \| x \|_{p,2} + \| S^* \| \; \| (I - P)Q_n x \|_{p,2} \\ &\leqq (\delta + 2 \sup \| Q_n \|_{p,2} \delta) \| x \|_{p,2}.\end{aligned}$$

Therefore range$(Q) =$ range(P) if δ is sufficiently small and Q is a $p, 2$-isomorphism from range(Q_n) onto its range.

To obtain the moreover assertion assume that we have $Q_1', Q_2', \ldots, Q_{n-1}'$ and construct Q as above for $B = [\text{range}(Q_{n-1}), x_n]$ and $C = [\text{range}(Q_{n-1}^*), x_n^*]$. Note that by Lemma 2.3 there is a $p, 2$-complemented subspace Y_n of the $\ker(Q) \cap \text{range}(Q^*)_\perp$ such that Y_n is $p, 2$-isomorphic to range(Q_{n-1}') and there is a $p, 2$-bounded projection Q' onto Y_n such that $\ker Q' \supset$ range(Q). Let $Q_n' = Q + Q'$. Q_n' is a projection satisfying $Q_n' b = b$ for all $b \in B$ and $Q_n'^* c = c$ for all $c \in C$. Also

$$\begin{aligned}\text{range}(Q_n' - Q_m') &= \text{range}(Q - Q_m') \oplus Y_n \\ &\sim (\text{range}(Q - Q_m') \oplus \text{range}(Q_{n-1}'))_{p,2} \\ &\sim (\text{range}(Q - Q_m') \oplus \text{range}(Q_{n-1}' - Q_m') \oplus \text{range}(Q_m'))_{p,2} \\ &\sim (\text{range}(Q) \oplus \text{range}(Q_{n-1}' - Q_m'))_{p,2} \\ &\sim (X_p^{k(n)} \oplus X_p^{k(n-1,m)})_{p,2}\end{aligned}$$

which is $p, 2$-isomorphic to X_p. (Here $\sim$ denotes $p, 2$-isomorphic.) Q.E.D.

Observe that the argument above also proves the following variant of Proposition 2.6.

PROPOSITION 2.6′. *Suppose that X is a $p, 2$-complemented subspace of X_p*

and that (Q_n) is a sequence of $p, 2$-continuous finite rank projection on X such that $\lim_n \| Q_n x - x \|_{p,2} = 0$ *for all* $x \in X$. *Then there is a sequence of projections (Q'_n) on X such that*

(i) *Q'_n is $p, 2$-continuous for each n,*

(ii) $Q'_n Q'_m = Q'_{\min(n,m)}$ *for all n and m,*

(iii) $\lim_n \| Q'_n x - x \| = 0$ *for all* $x \in X$.

Moreover if range(Q_n) *is K $p, 2$-isomorphic to $X^{k(n)}_{p,w(n)}$, for each n, then for any $\varepsilon > 0$ the sequence (Q'_n) may be chosen such that* range$(Q'_n - Q'_m)$ *is* $K(1+\varepsilon)$ *$p, 2$-isomorphic to $X^{k(n,m)}_{p,w(n,m)}$.*

3. Applications

In this section we will show that l_p and X_p are primary without using the decomposition method. We have completed all of the work except for some minor technicalities. First we will consider the case of l_p.

THEOREM 3.1 (Pelczynski). *If X is a complemented subspace of l_p and X has a complemented subspace isomorphic to l_p, then X is isomorphic to l_p.*

PROOF. We use the results of Section 2 but for the case where $|\cdot|_2$ is the same as the $|\cdot|_p$ norm. (This is essentially the case of [JRZ].) By Proposition 2.6, X has an FDD (X_i) such that for each $i < j$, $\Sigma^j_{n=i} X_i$ is isomorphic to $l_p^{n(i,j)}$. According to Theorem 1.1, X has a UFDD (Z_i) in which each Z^i is isomorphic to a block of the FDD (X_i) of X. Moreover, the UFDD in this case is isomorphically an l_p sum. Therefore X is isomorphic to an l_p sum of l_p^n's, which is, of course, isomorphic to l_p. Q.E.D.

THEOREM 3.2. ([JO]). *If X is a complemented subspace of X_p and X has a complemented subspace isomorphic to X_p, then X is isomorphic to X_p.*

PROOF. This time we will use the results of Section 2 for the case where $|\cdot|_2$ is the L_2 norm and $|\cdot|_p$ is the L_p norm. As in [JO] we may assume that X_p is isomorphically embedded in L_p and that X is $p, 2$-complemented in X_p. Because X_p is isomorphic to its square we may replace X_p by $X_p \oplus W$ and X by $X \oplus W$, where W is another copy of X_p in $L_p[1, 2)$. In this way we may assume that X is $p, 2$-complemented in X_p and contains a $p, 2$-complemented subspace which is $p, 2$-isomorphic to X_p. By Proposition 2.6, X has an FDD (X_i) such that for each $i < j$, $\Sigma^j_{n=i} X_i$ is uniformly $p, 2$-isomorphic to $X_p^{n(i,j)}$. According to Theorem 1.1, X has a UFDD (Z_i) in which each Z_i is $p, 2$-isomorphic to a block of the FDD (X_i) of X. Moreover, the UFDD in this case is a p, 2-sum. Therefore X is isomorphic to X_p. Q.E.D.

In [AEO] it was shown that if $X_p = Y \oplus Z$ then either Y or Z contains a complemented subspace isomorphic to X_p. Therefore we get

COROLLARY 3.3 [JO]. *X_p is primary.*

Note that if we use Proposition 2.6′ in the second part of the argument above and the isomorphic classification of the spaces $X_{p,w}$, [R], we get:

PROPOSITION 3.4. *If X is a p, 2-complemented subspace of $X_{p,w}$ and there is a constant $K < \infty$ such that every finite dimensional subspace Y of X is contained in a finite dimensional subspace Z of X which is K p, 2-complemented in X and K p, 2-isomorphic to $X_{p,w(Z)}^{n(Z)}$, then X is p, 2-isomorphic to l_p, l_2, $l_p \oplus l_2$, or X_p.*

Theorem 2.1 is an immediate consequence of this proposition. Also note that if we use Proposition 2.5 in addition we get the following localization of Theorem 3.2:

PROPOSITION 3.5. *If X is a p, 2-complemented subspace of $X_p = X_{p,w}$ and there is a constant $K < \infty$ such that for every n, X contains a finite dimensional subspace Z which is K p, 2-complemented in X and K p, 2-isomorphic to $X_{p,w}^{n,1}$, then X is p, 2-isomorphic to $X_{p,w}$.*

Proposition 3.4 seems to give an approach to the problem of classifying isomorphically the complemented subspaces of X_p. Unfortunately a significant improvement in the techniques of construction of FDD's used here is needed. Note that Proposition 3.5 is false if the subspaces Z are merely required to be isomorphic to $X_{p,w}^{n,1}$ because l_p is isomorphic to $(\Sigma \oplus X_{p,w}^{n,1})_p$. Also notice that it follows from the arguments in the previous section that there is a sequence of commuting finite rank projections on l_p, (P_n), such that range(P_n) is isomorphic to $X_{p,w}^{k(n),1}$ (uniformly in n) and (P_n) converges in the strong operator topology to the identity. The same thing could be done with other $\mathscr{L}_p$ spaces. Thus without some special restriction on the operators the finite dimensional information is of little value in determining the isomorphic nature of the space.

The same arguments we have used above are applicable to certain complemented subspaces of rearrangement invariant sequence spaces. The main observation is that if l_p is such a space with Boyd indices p_1 and q_1 and $p < p_1$ and $q > q_1$, then the disjoint summability criterion is satisfied for the class of simultaneously weak type (p, p) and (q, q) operators. In order to state our results succinctly let us say that an operator on a complemented subspace of l_p is weak type (p, p) if the composition of the projection with the operator is weak type (p, p). Using the argument given for X_p, we get the following results.

PROPOSITION 3.6. *Suppose that* l_p *is a rearrangement invariant sequence space with Boyd indices* p_1 *and* q_1 *and that* P *is a projection on* l_p *which is weak type* (p, p) *and* (q, q) *for some* $p < p_1$ *and* $q > q_1$. *If there is a constant* K *such that for every* $n \in \mathbf{N}$ range(P) *contains a subspace* Z *which is weak type* (p, p) *and* (q, q) *complemented in* l_p *and is weak type* (p, p) *and* (q, q) *isomorphic to* l_p^n *with constants bounded by* K, *then* range(P) *is isomorphic to* l_p.

COROLLARY 3.7. *Suppose that* l_p *is a rearrangement invariant sequence space with Boyd indices* p_1 *and* q_1. *If* P *is a projection on* l_p *which is weak type* (p, p) *and* (q, q) *for some* $p < p_1$ *and* $q > q_1$, *then either* range(P) *or* range($I - P$) *is isomorphic to* l_p.

PROOF. The lemma of Casazza and Lin [CL] shows that either range(P) or range($I - P$) contains a complemented subspace isomorphic to l_p. Without loss of generality assume that it is range(P). Because $l_p \oplus l_p$ is isometric to l_p (the direct sum is identified with $l_p(2\mathbf{N}) \oplus l_p(2\mathbf{N} + 1)$), we may replace P by $P \oplus I$ to get a weak type (p, p) and (q, q) projection with range containing a weak type (p, p) and (q, q) complemented subspace which is weak type (p, p) and (q, q) isomorphic to l_p. Proposition 3.6 completes the proof. Q.E.D.

QUESTION. If X is a complemented subspace of l_p, is there a subspace Z of l_p which is isomorphic to X and is weak type (p, p) and (q, q) complemented in l_p?

It is conceivable that a complemented subspace of l_p can always be repositioned to be the range of a weak type (p, p) and (q, q) projection. If this were the case then Corollary 3.7 would imply that l_p is primary.

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